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ARBITRARY ℓ-SOLUTIONS OF THE SCHRÖDINGER EQUATION FOR SUPERPOSITION OF ATTRACTIVE RADIAL AND MODIFIED COULOMB POTENTIAL WITH SPIN-ORBIT INTERACTION (SACS)

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Abstract

In this study, we obtained the bound-state solutions of the radial part of the three-dimensional Schrödinger equation for a superposition of an attractive radial potential and a modified Coulomb potential with spin–orbit interaction (SACS), using the Nikiforov–Uvarov (NU) method for arbitrary quantum states. The Greene–Aldrich approximation was employed to treat the inverse and inverse-square terms in the resulting effective potential. We also derived the corresponding normalized wave functions, expressed in terms of Jacobi polynomials, for a single particle moving in this potential. Furthermore, we analytically calculated the energy eigenvalues for various quantum states, explicitly demonstrating the effects of spin–orbit coupling in different excited states. In addition, two special cases of the potential and their corresponding energy eigenvalues were obtained. Our results show excellent agreement with those reported in the existing literature.

Keywords: Schrödinger equation; Nikiforov–Uvarov method; energy eigenvalues; attractive radial potential; spin–orbit interaction; modified Coulomb potential

1.0 Introduction

The analytical solution of the Schrödinger equation remains a cornerstone of quantum mechanics, as it provides exact insights into the spectral and wave-function properties of quantum systems. Exact and approximate solutions are particularly valuable for understanding bound states, degeneracy structures, and symmetry properties of physical systems, as well as for benchmarking numerical and perturbative approaches [1, 2].

Over the years, several analytical techniques have been developed to solve the Schrödinger equation for different potential models. Prominent among these are the Nikiforov–Uvarov method [3, 4], supersymmetric quantum mechanics, factorization methods, asymptotic iteration methods, and point canonical transformations [5, 6]. These methods have proven effective in obtaining exact or quasi-exact solutions for a wide class of central and non-central potentials, enabling the determination of energy spectra and normalized eigenfunctions.

Attractive radial potentials represent a general class of spherically symmetric potentials characterized by an overall attractive interaction. Such potentials are commonly employed to model short-range forces in quantum systems, particularly when deviations from idealized Coulombic behavior are present [7]. In atomic and molecular physics, attractive radial potentials are used to describe effective electron–nucleus interactions, especially in screened or many-body environments. In nuclear physics, they appear naturally in mean-field descriptions of nucleon motion inside finite nuclei [8].

Analytical solutions for attractive radial potentials provide valuable information about bound states, radial probability distributions, and the influence of angular momentum barriers. These studies are essential for understanding confinement mechanisms and localization effects in quantum systems.

The Coulomb potential is one of the most fundamental interaction models in quantum mechanics and admits an exact analytical solution within the Schrödinger framework. Its solution forms the basis for the quantum theory of hydrogen-like atoms and has been extensively discussed in standard quantum mechanics texts [9]. The Coulomb problem yields discrete energy levels and eigenfunctions that depend explicitly on the principal and orbital quantum numbers.

Despite its exact solvability, the Coulomb potential alone often provides an idealized description. In realistic systems, screening effects, relativistic corrections, and additional short-range interactions necessitate the inclusion of supplementary potential terms. Consequently, studies involving modified or superposed Coulomb potentials continue to attract significant attention [10].

The superposition of an attractive radial potential with the Coulomb potential in the presence of spin–orbit interaction offers a more realistic framework for modeling quantum interactions involving both long-range and short-range forces. Such composite potentials are particularly relevant in exotic atoms, quarkonium systems, and effective interaction models in nuclear and particle physics [11].

Analytical investigations of these superposed potentials reveal modified energy spectra and wave-function structures compared to the pure Coulomb case [12]. The presence of additional radial terms often leads to energy shifts, altered degeneracy patterns, and enhanced flexibility in fitting experimental data. These features make superposed potentials highly attractive for phenomenological modeling.

The spin–orbit interaction arises from the coupling between a particle’s spin and its orbital angular momentum and is responsible for fine-structure splitting in atomic spectra.

Within the non-relativistic approximation, the spin–orbit term is typically introduced as an additional effective potential proportional to the scalar product of the spin and orbital angular momentum operators [13].

The inclusion of spin–orbit coupling significantly enriches the physical content of the Schrödinger equation by lifting degeneracies and introducing spin-dependent effects. In atomic physics, it explains fine-structure corrections, while in nuclear physics it plays a crucial role in the shell structure and magic numbers of nuclei [12]. In condensed matter physics, spin–orbit interaction underpins many modern applications, including spintronics and topological materials [14].

Analytical solutions of the Schrödinger equation for superposed attractive radial and Coulomb potentials with spin–orbit interaction have broad applications across multiple fields. In atomic physics, they contribute to high-precision modeling of spectral lines and fine-structure effects. In nuclear physics, they provide insight into single-particle motion and spin–orbit splitting in mean-field potentials. In condensed matter physics, similar models are employed in the study of quantum dots, impurity states, and spin-dependent transport phenomena [15].

The superposition of attractive radial and Coulomb potentials, augmented by spin–orbit interaction, represents a mathematically rich and physically relevant system. Continued analytical investigations in this area remain essential for advancing theoretical understanding and supporting applications in atomic, nuclear, and condensed matter physics.

2.0 REVIEW OF THE NU METHOD

We present a brief overview of the NU method in this section; a detailed presentation of this method could be obtained from ref. [16]. This method entails finding a solution to a hyper-geometric type second order differential equation:

$$\psi''(z)+\frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z)+\frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z)=0 \qquad 1$$

Where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials of the second degree. $\tilde{\tau}(z)$ is a first degree polynomial [19]. A proposed solution to (1) is given as:

$$\psi(z)=\phi(z)y(z) \qquad 2$$

This result to a hyper geometric – type equation of the form;

$$\sigma(z)y''(z)+\tau(z)y'(z)+\lambda y(z)=0 \qquad 3$$

Part of the solution of (1) is given in (2) as $\phi(z)$ is the solution to another differential equation of the form given by;

$$\sigma(z)\phi'(z)-\pi(z)\phi(z)=0 \qquad 4$$

Where

$$\tau(z)=\tilde{\tau}(z)+2\pi(z) \qquad 5$$

And λ in (3) is defined as;

$$\lambda=\lambda_n=-n\tau'(z)+\frac{n(n-1)}{2}\sigma''(z)=0(n=0,1,2,3,...) \qquad 6$$

The term $\tau(z)$ is a polynomial, its first derivative $\tau'(z)$ must be negative for a proper solution of the hyper-geometric type differential equation. The function $y(z)$ as stated in (2) is the hyper geometric – type wave function obtained from the Rodrigues relation:

$$y_n(z)=\frac{B_n(z)d^n}{\rho(z)dz^n}[\sigma^n(z)\rho(z)] \qquad 7$$

Where B_n is related to the normalization constant, and $\rho(z)$ is defined as:

$$\frac{d}{dz}[\sigma(z)\rho(z)] = \tau(z)\rho(z) \qquad 8$$

Also the function $\pi(z)$ is a first degree polynomial is defined as:

$$\pi(z) = \frac{\sigma'(z)-\tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z)-\tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)} \qquad 9$$

K in (9) is related to (6) and the first derivative of $\pi'(z)$ as thus:

$$\lambda = k + \pi'(z) \qquad 10$$

The value of k is obtained by equating the terms under the square root sign in (9) to zero. By solving (6) and (10), we derive the energy eigenvalue equation.

3. SOLUTION OF SCHRÖDINGER EQUATIION FOR SACS

The energy spectrum $E_{n\ell j}$ of a single particle system obtained by solving the radial component of the 3-dimensional Schrödinger wave equation given as:

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \frac{2\mu}{\hbar^2}\left[E_{n\ell j} - V_{eff}(r)\right]R(r) = 0 \qquad 11$$

where μ is the reduced mass of the system, $E_{n\ell j}$ is the energy spectra, $\hbar$ is the Planck's constant, n, ℓ and j are the principal, orbital and total momentum quantum number respectively. Let define a new variable be defined as:

$$Z = e^{-2\alpha r} \qquad 12$$

The derivatives of the function $R(r)$ in (11) are presented in terms of the new variable z, thereby transforming to a hyper-geometric second order differential equation, hence $R(r)$ is redefined as $\varphi(z)$. Therefore, an approximate solution of the Schrödinger equation for the effective potential applied in the present work.

Using the variable as defined by (12), (11) is transformed and the Schrödinger equation becomes as state below;

$$\frac{d^2\varphi(z)}{dz^2} + \frac{1}{z}\frac{d\varphi(z)}{dz} + \frac{1}{\alpha^2 z^2}\left[\frac{2\mu E}{\hbar^2} - \frac{2\mu}{\hbar^2}\left(V_{eff}\right)\right]\varphi(z) = 0 \qquad 13$$

The effective potential here is the SACS potential which is a superposition of Attractive radial, adjusted Coulomb, spin-orbit interaction and the centrifugal term. The Attractive radial potential is given as;

$$V(r) = \frac{Ae^{-2\alpha r} + B + Ce^{2\alpha r}}{e^{2\alpha r}(1-e^{-2\alpha r})^2} \qquad 14$$

Where;

$$A = \frac{\alpha^2}{4}$$

$$B = A(\lambda - 8)$$

$$C = A\left(4 - \lambda\right)$$

Where A, B, C and α are all potential parameters.

Given that A and α are real and $\alpha > \frac{1}{2}, 4 < \lambda < 8$

The Attractive radial potential can rewritten as

$$V_{att}(r) = \frac{Ae^{-4\alpha r} + Be^{-2\alpha r} + C}{\left(1-e^{-2\alpha r}\right)^2} \qquad 15$$

The modified Coulomb potential is given as;

$$V_c(r) = \frac{e^2}{R_o}\left[3 - \left(\frac{r}{R_o}\right)^2\right] \qquad 16$$

The spin-orbit interaction term is given by (17)

$$V_{LS}(r) = V_{LS}(0)\left(\frac{R_o}{\hbar}\right)^2 \frac{1}{r}\left[\frac{dV_{att}(r)}{dr}\right]\vec{L}.\vec{S} \qquad 17$$

Applying the approximation scheme report by [17, 18] as stated in (18), the spin orbit interaction is presented as (19);

$$\frac{1}{r} \approx \frac{\alpha}{\left(1-e^{-2\alpha r}\right)}$$

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\left(1-e^{-2\alpha r}\right)^2} \qquad 18$$

$$\frac{1}{r^2} \approx \frac{\alpha^2 e^{-\alpha r}}{\left(1-e^{-\alpha r}\right)^2}$$

18

$$V_{LS}(s) = \frac{1}{2} V_{LS}(0) \frac{r_o^{\,2}}{\hbar^2} \left[\frac{4A\alpha^3 e^{-4\alpha r}(e^{-2\alpha r}-1) + 2B\alpha^2 e^{-4\alpha r}(e^{-2\alpha r}-1)}{(1-e^{-2\alpha r})^5} \frac{+4C\alpha^2 e^{-2\alpha r}(e^{-2\alpha r}-1)}{} \right] \vec{L}.\vec{S} \qquad 19$$

Where $\vec{L}.\vec{S} = \frac{\hbar^2}{2}\left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right)$

The effective potential SACS is given as (20);

$$V_{eff} = \left[\begin{array}{l} \dfrac{Ae^{-4\alpha r} + Be^{-2\alpha r} + c}{\left(1-e^{-2\alpha r}\right)^2} + 4V_{LS}(0) r_o^2 \alpha^2 \left[\dfrac{Ae^{-4\alpha r}(e^{-2\alpha r}-1)}{(1-e^{-2\alpha r})^4} \right. \\ \left. + \dfrac{Ae^{-4\alpha r}}{(1-e^{-2\alpha r})^4} + \dfrac{{}^1\!/_2 Be^{-2\alpha r}(1-e^{-2\alpha r})}{(1-e^{-2\alpha r})^4} + \dfrac{Be^{-2\alpha r}}{(1-e^{-2\alpha r})^4} + \dfrac{Ce^{-2\alpha r}}{(1-e^{-2\alpha r})^4} \right] \vec{L}.\vec{S} \\ + \dfrac{3e^2}{\pi\varepsilon_0 R_o} - \dfrac{e^2}{\pi\varepsilon_0 R_o^3} \dfrac{(1-e^{-2\alpha r})^2}{\alpha^2} + \dfrac{\hbar^2 \alpha^2 e^{-2\alpha r} \ell(\ell+1)}{2\mu(1-e^{-2\alpha r})^2} \end{array} \right] \qquad 20$$

Substituting (20) into (13), the Schrödinger equation becomes;

$$\frac{d^2\varphi(z)}{dZ^2} + \frac{1}{Z}\frac{d\varphi(z)}{dZ} + \frac{1}{Z^2(1-Z)^2} \left[\begin{array}{l} \dfrac{\mu E}{2\alpha^2\hbar^2}(1-Z)^2 - \dfrac{\mu}{2\alpha^2\hbar^2}\left(AZ^2 + BZ + C\right) \\ -V_{LS}(0)\dfrac{\mu r_o^2}{\hbar^2}(AZ^2(1-Z)^{-1} + AZ(1-Z)^{-2} \\ + {}^1\!/_2 BZ(1-Z)^{-1} + BZ(1-Z)^{-2} + CZ(1-Z)^{-2})\vec{L}.\vec{S} \\ -\dfrac{3\mu e^2}{2\pi\varepsilon_0 R_o \alpha^2 \hbar^2}(1-Z)^2 + \dfrac{\mu e^2}{\hbar^2 \pi\varepsilon_0 R_o^3}\dfrac{(1-Z)^4}{\alpha^4} - Z\ell(\ell+1) \end{array} \right] \varphi(z) = 0 \qquad 21$$

Expanding the $(1 - Z)^n$ terms within the square bracket in power series and truncating the higher powers of Z, the Schrödinger equation is transformed to the hyper-geometric second order differential equation which is presented as (22) solvable by the NU method.

$$\frac{d^2\psi(z)}{dZ^2} + \frac{(1-Z)}{Z(1-Z)}\frac{d\psi(z)}{dZ} + \frac{1}{Z^2(1-Z)^2}\left(\Im Z^2 + \Re Z - \aleph\right)\varphi(z) = 0 \qquad 22$$

Where;

$$\Im=\left\{\begin{array}{l}-\frac{\mu E}{2\alpha^2\hbar^2}-\frac{\mu}{2\alpha^2\hbar^2}A-2V_{LS}(0)\frac{\mu r_o^2}{\hbar^2}\vec{L}.\vec{S}B-{}^3\!/_2\,\beta_2=V_{LS}(0)\frac{\mu r_o^2}{\hbar^2}\vec{L}.\vec{S}B\\ -2\beta_2=V_{LS}(0)\frac{\mu r_o^2}{\hbar^2}\vec{L}.\vec{S}C-\frac{3\mu e^2}{2\pi\varepsilon_0\alpha^2\hbar^2R_o}+6\frac{\mu e^2}{2\pi\varepsilon_0\alpha^4\hbar^2R_0^3}\end{array}\right\} \qquad 23$$

$$\Re=\left\{\begin{array}{l}2\frac{\mu E}{2\alpha^2\hbar^2}-\frac{\mu}{2\alpha^2\hbar^2}B-{}^3\!/_2\,V_{LS}(0)\frac{\mu r_o^2}{\hbar^2}\vec{L}.\vec{S}B-V_{LS}(0)\frac{\mu r_o^2}{\hbar^2}\vec{L}.\vec{S}C\\ +2\frac{3\mu e^2}{2\pi\varepsilon_0\alpha^2\hbar^2R_o}-4\frac{\mu e^2}{2\pi\varepsilon_0\alpha^4\hbar^2R_0^3}-\ell\left(\ell+1\right)\end{array}\right\} \qquad 24$$

$$-\aleph=\left\{\frac{\mu}{2\alpha^2\hbar^2}C-\frac{\mu E}{2\alpha^2\hbar^2}+\frac{3\mu e^2}{2\pi\varepsilon_0\alpha^2\hbar^2R_o}-\frac{\mu e^2}{2\pi\varepsilon_0\alpha^4\hbar^2R_0^3}\right\} \qquad 25$$

Comparing (22) to (1), the requisite polynomials are obtained as thus;

$$\tilde{\tau}(z)=1-Z,\qquad \sigma(z)=Z\left(1-Z\right),\qquad \sigma^2(z)=Z^2\left(1-Z\right)^2$$
$$\sigma(z)=\Im Z^2+\Re Z-\aleph \qquad 26$$

Substituting the polynomials defined in (26) into (9) we have;

$$\pi(z)=\frac{-Z}{2}\pm\sqrt{\left(\chi-K\right)Z^2+\left(\Re+K\right)Z+\aleph} \qquad 27$$

Where;

$$\chi=\frac{1}{4}+\Im \qquad 28$$

According to the NU method K in (27) is obtained by equating the discriminant of the quadratic equation under the square root sign of (27) to zero. The resultant quadratic equation in K is solved, and the root of the equation is determined as;

$$K=-\left(\Re+2\aleph\right)\pm2\sqrt{\aleph}\sqrt{\chi+\Re+\aleph} \qquad 29$$

Substituting the negative solution of K into (27), we have;

$$\pi(z)=\frac{-z}{2}\pm\left(\sqrt{\aleph}+\sqrt{\chi+\Re+\aleph}\right)z-\sqrt{\aleph} \qquad 30$$

Upon inserting (30) into (5) another polynomial is obtained

$$\tau(z)=1-2z-2\left(\sqrt{\aleph}+\sqrt{\chi+\Re+\aleph}\right)z-2\sqrt{\aleph} \qquad 31$$

Taking the first derivative of $\tau(z)$ with respect to z as expressed in (32) we have;

$$\tau^{|}(z)=-2-2\sqrt{\aleph}-2\sqrt{\chi+\Re+\aleph} \qquad 32$$

Taking the first derivative of (30) and using (29), (32) as well as (26), λ and λ_n are obtained explicitly as;

$$\lambda=\frac{-1}{2}-\sqrt{\aleph}-\sqrt{\chi+\Re+\aleph}-\Re-2\aleph-2\sqrt{\aleph}\sqrt{\chi+\Re+\aleph} \qquad 33$$

$$\lambda_n = n^2 + n + 2n\sqrt{\aleph} + 2n\sqrt{\chi + \Re + \aleph} \qquad 34$$

By equating (33) and (34), we solve for the term $\aleph$;

$$\aleph = \left[\frac{-\frac{1}{2}\left[\left(n+\frac{1}{2}+\sqrt{\gamma}\right)^2 + \frac{1}{4} + \beta\right]}{\left(n+\frac{1}{2}+\sqrt{\gamma}\right)}\right]^2 \qquad 35$$

Where

$$\begin{aligned}\gamma &= \chi + \Re + \aleph \\ \beta &= -\frac{1}{4} - \chi + \aleph\end{aligned} \qquad 36$$

Putting (23), (24), and (25) into (35) and (36) we obtain the energy eigenvalues E of the SACS potential as given by (37).

$$E = C + \frac{3e^2}{\pi\varepsilon_0 R_o} - \frac{e^2}{\pi\varepsilon_0\alpha^2 R_o^3} - \frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\mu}{2\alpha^2\hbar^2}(A+B+C)+\left(2A+3B+3C\right)\frac{V_{LS}(0)\mu r_o^2}{\hbar^2}\left(j(j+1)-\ell(\ell+1)-\frac{3}{4}\right)-\frac{3\mu e^2}{2\pi\varepsilon_0\alpha^4\hbar^2R_o^3}+\ell(\ell+1)}\right)^2}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\mu}{2\alpha^2\hbar^2}(A+B+C)+\left(2A+3B+3C\right)\frac{V_{LS}(0)\mu r_o^2}{\hbar^2}\left(j(j+1)-\ell(\ell+1)-\frac{3}{4}\right)-\frac{3\mu e^2}{2\pi\varepsilon_0\alpha^4\hbar^2R_o^3}+\ell(\ell+1)}\right)} + \frac{\frac{\mu}{2\alpha^2\hbar^2}(-A+C)-\left(2A+\frac{3}{2}B+2C\right)\frac{V_{LS}(0)\mu r_o^2}{\hbar^2}\left(j(j+1)-\ell(\ell+1)-\frac{3}{4}\right)+\frac{5\mu e^2}{2\pi\varepsilon_o\alpha^4\hbar^2R_o^3}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\mu}{2\alpha^2\hbar^2}(A+B+C)+\left(2A+3B+3C\right)\frac{V_{LS}(0)\mu r_o^2}{\hbar^2}\left(j(j+1)-\ell(\ell+1)-\frac{3}{4}\right)-\frac{3\mu e^2}{2\pi\varepsilon_0\alpha^4\hbar^2R_o^3}+\ell(\ell+1)}\right)}\right]^2 \qquad 37$$

The corresponding eigen function is given as;

$$\varphi(z)=\left[\begin{array}{l} z^{\sqrt{\frac{-\mu E}{2\alpha^2\hbar^2}+\frac{\mu C}{2\alpha^2\hbar^2}}}. \\ (1-z)^{\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\mu}{2\alpha^2\hbar^2}(A+B+C)+(2A+3B+3C)\frac{V_{LS}(0)\mu r_o^2}{\hbar^2}\left(j(j+1)-\ell(\ell+1)-\frac{3}{4}\right)+\ell(\ell+1)}\right)}. \\ P_n^{\left(2\sqrt{\frac{-\mu E}{2\alpha^2\hbar^2}+\frac{\mu C}{2\alpha^2\hbar^2}+\ell(\ell+1)}\right)}. \\ P_n^{\left(2\sqrt{\frac{1}{4}+\frac{\mu}{2\alpha^2\hbar^2}(A+B+C)+(2A+3B+3C)\frac{V_{LS}(0)\mu r_o^2}{\hbar^2}\left(j(j+1)-\ell(\ell+1)-\frac{3}{4}\right)+\ell(\ell+1)}\right)}(1-2Z) \end{array}\right] \tag{38}$$

4. RESULTS AND DISCUSSION

Two special cases were be obtained from the solution of the Schrödinger equation for SACS potential.

First, (39) presents a special case of the eigenvalues of the SACS potential suitable for a charge-less particle. As such e = 0 and the Coulomb term varnish, thus (39) is the energy spectrum of attractive radial potential with spin-orbit interaction.

$$E=C-\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\mu}{2\alpha^2\hbar^2}(A+B+C)+(2A+3B+3C)\frac{V_{LS}(0)\mu r_o^2}{\hbar^2}\left(j(j+1)-\ell(\ell+1)-\frac{3}{4}\right)+\ell(\ell+1)}\right)^2}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\mu}{2\alpha^2\hbar^2}(A+B+C)+(2A+3B+3C)\frac{V_{LS}(0)\mu r_o^2}{\hbar^2}\left(j(j+1)-\ell(\ell+1)-\frac{3}{4}\right)+\ell(\ell+1)}\right)}+\frac{\frac{\mu}{2\alpha^2\hbar^2}(-A+C)-\left(2A+\frac{3}{2}B+2C\right)\frac{V_{LS}(0)\mu r_o^2}{\hbar^2}\left(j(j+1)-\ell(\ell+1)-\frac{3}{4}\right)}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\mu}{2\alpha^2\hbar^2}(A+B+C)+(2A+3B+3C)\frac{V_{LS}(0)\mu r_o^2}{\hbar^2}\left(j(j+1)-\ell(\ell+1)-\frac{3}{4}\right)+\ell(\ell+1)}\right)}\right]^2 \tag{39}$$

The second of special cases obtained from (37) is a case where $r_o = 0$ which is a potential parameter of the spin – orbit interaction. The second case gives the energy spectrum for the attractive radial potential only, as given by (40).

$$E=C-\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\mu}{2\alpha^2\hbar^2}(A+B+C)+\ell(\ell+1)}\right)^2+\frac{\mu}{2\alpha^2\hbar^2}(-A+C)}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\mu}{2\alpha^2\hbar^2}(A+B+C)+\ell(\ell+1)}\right)}\right]^2 \tag{40}$$

The numerical results of (37) for arbitrary ℓ is given on Table 1. The following parameter where used for the calculation; $\mu=\hbar=C=\varepsilon=e=R_o=V_{LS}=1$, $\alpha=0.75,\lambda=6$ and $r_o=0.03$. The numerical results of (40) in terms of its relationship with the potential parameter α is given on Table 2. The numerical results is arrived at using the following parameters; $\mu = \hbar = C = 1$ and $\lambda = 6$.Table 1: Numerical values of the eigenvalues of SACS for arbitrary ℓ.

n	ℓ	j	Energy
2	1	½	-4.023520593
		3/2	-3.896372163
3	1	½	-6.036557611
		3/2	-5.843229770
	2	3/2	-9.980517439
		5/2	-9.886146246
4	1	½	-8.639185004
		3/2	-8.385683714
	2	3/2	-13.41204155
		5/2	-13.28792656
	3	5/2	-17.94226955
		7/2	-17.82951766
5	1	½	-11.81538186
		3/2	-11.50280057
	2	3/2	-17.41000510
		5/2	-17.25669423
	3	5/2	-22.57621829
		7/2	-22.43697621
	4	7/2	-28.11567240
		9/2	-27.98119497

Table 2: Numerical results of (40) showing the relationship between energy and potential parameter α

State	α	Energy	Onate, et al (2018).
2p	0.55	-0.57241325	-0.5905882
	0.65	-0.79946279	-0.8248712
	0.75	-1.06440481	-1.0982013
3p	0.55	-1.28562500	-1.2782588
	0.65	-1.79562500	-1.7853359
	0.75	-2.39062500	-2.3769265
3d	0.55	-1.33331254	-1.3610387
	0.65	-1.86222991	-1.9009549
	0.75	-2.47930018	-2.5308527
4p	0.55	-2.30367737	-2.2687305
	0.65	-3.21753286	-3.1687227

	0.75	-4.28369760	-4.2187137
4d	0.55	-2.36791757	-2.3808515
	0.65	-3.30725677	-3.3253215
	0.75	-4.40315251	-4.4272032
4f	0.55	-2.39429406	-2.4253796
	0.65	-3.34409667	-3.3875137
	0.75	-4.45219971	-4.5100034

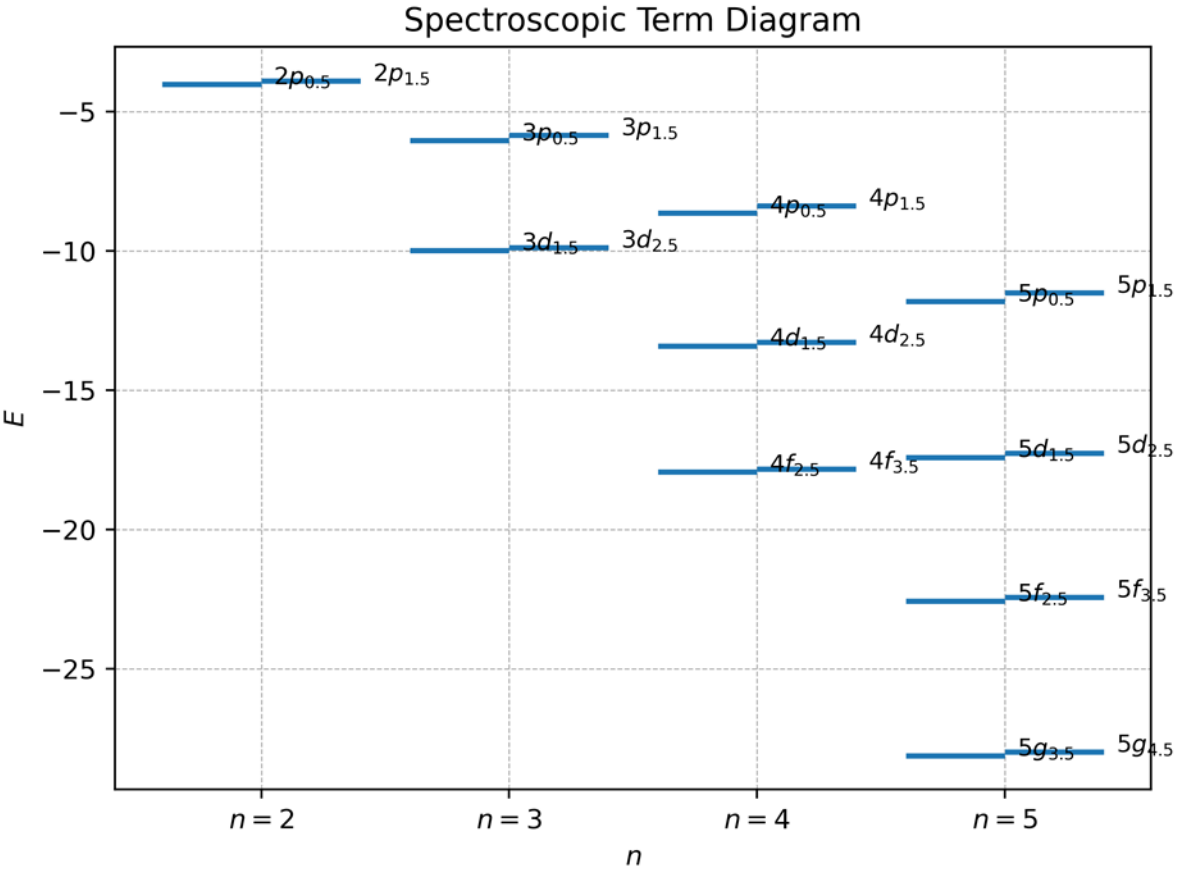

Fig. 1: Chart showing energy splitting in spectroscopic terms

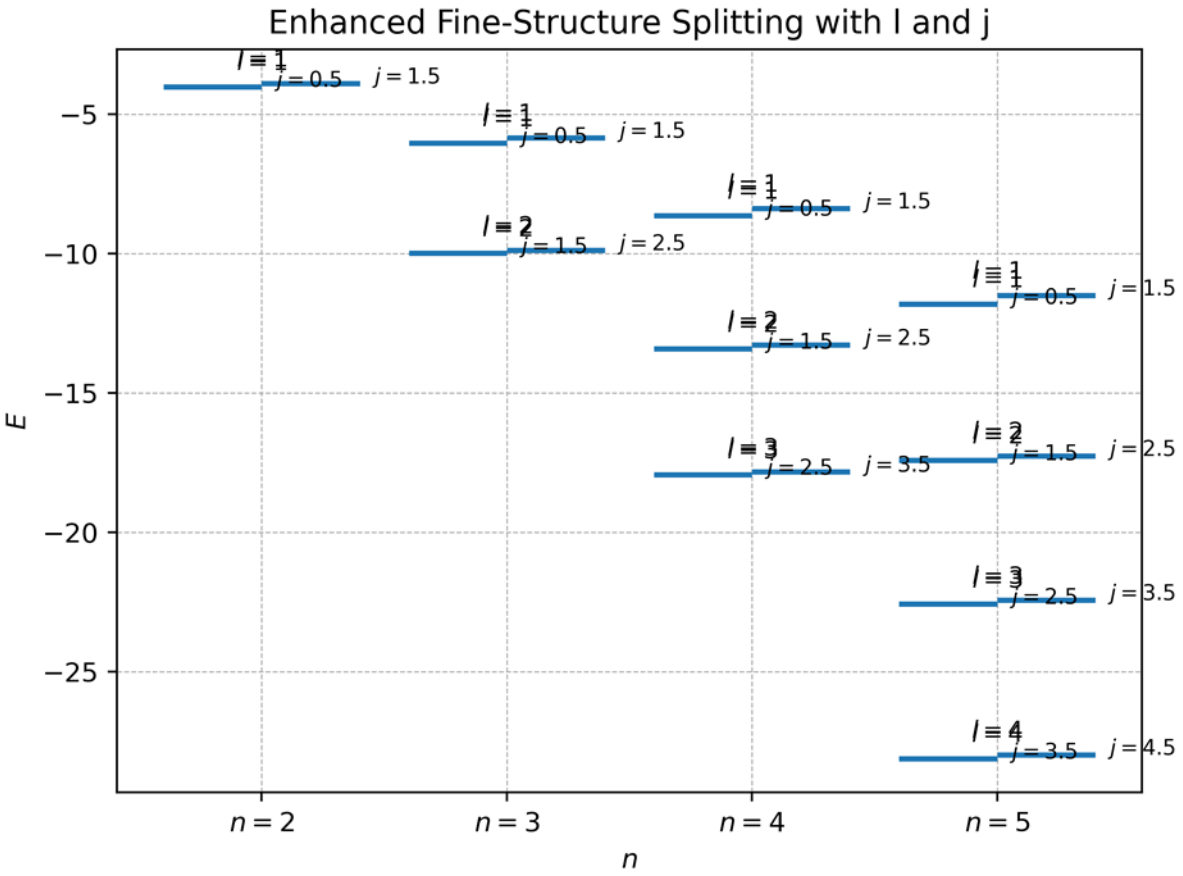

Fig. 2: Graphical presentation of energy splitting due spin orbit interaction

Table 1 gives the numerical results of energy eigenvalues given by (37) for a particle in a system defined by the superposition of attractive radial, modified Coulomb potential plus spin – orbit interaction (SACS). It can be observed on Table 1 that the energy sub – levels as specified by the respective values of ℓ splits as a results of the spin – orbit interaction, except for the $\ell=0$ state. The splitting of the orbital energy levels is according to the $j=\ell\pm{}^{1}\!/_{2}$ scheme for each ℓ as showing in Fig. 2. Fig. 1 present the energy splitting in spectroscopic terms. Interestingly, the non- relativistic S state energy for the attractive potential does not exist. Hence the attractive potential within the non – relativistic quantum mechanics may not aptly apply to the determination of the ground state energy of a particle. Table 2 presents the numerical results of (40), as it varies with the potential parameter α. The results shows that the energy increases negatively with increase in α.

In order to measure the level of accuracy of the analytical solution as obtained for the SACS potentials in the present work, the numerical results on Table 2 is compared to similar results presented by Onate, et al (2018) who solved the Schrödinger equation for the attractive radial potential only. This comparison is done by setting the spin orbit interaction and the Coulomb terms in the SACS potential to zero. Thus reducing the energy spectrum for SACS given in (37) to (40) which is the energy spectrum for the radial attractive potential only which was solved by Onate, et al (2028) using a different analytical method. The comparison shows minimum percentage relative error of 0.7% for 3p($\alpha=0.55$) state, and a maximum percentage relative error of 18% for 4d(

$\alpha = 0.75$) state. For all other states the percentage relative error lies between 0.9% and 5.0%. This implies that our analytical solution of the Schrödinger equation for the SACS potential is not only accurate but in agreement with results obtained from related study.

5.0 CONCLUSION

The SACS potential offers a physically more realistic and broader quantum mechanical model for studying non relativistic quantum mechanical systems, especially when considering long and short range interactions. This potential model can be suitably applied in areas of particle physics, nuclear physics and quarkonium systems.

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